REMARKS ON A SCALAR CURVATURE RIGIDITY THEOREM OF BRENDLE AND MARQUES

GRAHAM COX, PENGZI MIAO[1], AND LUEN-FAI TAM[2]

ABSTRACT. We give an improvement of a scalar curvature rigidity theorem of Brendle and Marques regarding geodesic balls in $\mathbb{S}^n$. The main result is that Brendle and Marques' theorem holds on a geodesic ball larger than that specified in [2].

1. INTRODUCTION

In a recent paper [2], Brendle and Marques proved the following theorem on scalar curvature rigidity of geodesic balls in the standard n-dimensional sphere $\mathbb{S}^n$.

Theorem 1.1 (Brendle and Marques [2]). *Let $\Omega = B(\delta) \subset \mathbb{S}^n$ be a closed geodesic ball of radius δ with*

$$\cos\delta \geq \frac{2}{\sqrt{n+3}}. \tag{1.1}$$

Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Suppose g is another metric on Ω with the properties:

- *$R(g) \geq R(\bar{g})$ at each point in Ω*
- *$H(g) \geq H(\bar{g})$ at each point on $\partial\Omega$*
- *g and $\bar{g}$ induce the same metric on $\partial\Omega$*

where $R(g)$, $R(\bar{g})$ are the scalar curvature of g, $\bar{g}$, and $H(g)$, $H(\bar{g})$ are the mean curvature of $\partial\Omega$ in (Ω, g), $(\Omega, \bar{g})$. If $g - \bar{g}$ is sufficiently small in the C^2-norm, then $\varphi^*(g) = \bar{g}$ for some diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \mathrm{id}$.

Theorem 1.1 is an interesting rigidity result for domains in $\mathbb{S}^n$ because the corresponding statement is false for $\delta = \frac{\pi}{2}$, which follows from the counterexample to Min-Oo's conjecture ([6]) constructed by Brendle, Marques and Neves in [3]. For an account of the connection of Theorem

2010 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C24.

[1]Research partially supported by Australian Research Council Discovery Grant #DP0987650 and by a 2011 Provost Research Award of the University of Miami.

[2]Research partially supported by Hong Kong RGC General Research Fund #CUHK 403011.

1.1 to other rigidity phenomena involving scalar curvature, readers are referred to the recent survey [1] by Brendle.

In this paper, we provide an improvement of Theorem 1.1 by showing that Theorem 1.1 is still valid on geodesic balls strictly *larger* than those specified by (1.1). Precisely, we prove that condition (1.1) in Theorem 1.1 can be replaced by either one of the following weaker conditions:

(**a**) $\cos\delta > \zeta$, *where* ζ *is the positive constant given by*

$$\zeta^2 = \frac{4(n+4) - 4\sqrt{2n-1}}{n^2+6n+17}.$$

(**b**) $\cos\delta > \cos\delta_0$, *where* δ_0 *is the unique zero of the function*

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}$$

where $\alpha(\delta) = \frac{(n+1)}{8n}\left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1}$ *and* $\mu(\delta)$ *is the first nonzero Neumann eigenvalue of* $B(\delta)$*. In particular,* δ_0 *satisfies*

$$(\cos\delta_0)^2 < \frac{7n-1}{2n^2+5n-1}. \tag{1.2}$$

We compare the conditions (**a**) and (**b**). It follows from (1.2) that δ_0 in (**b**) satisfies

$$\limsup_{n\to\infty} \frac{(\cos\delta_0)^2}{\frac{4}{n+3}} \le \frac{7}{8}, \tag{1.3}$$

while in (**a**) one has

$$\lim_{n\to\infty} \frac{\frac{4(n+4)-4\sqrt{2n-1}}{n^2+6n+17}}{\frac{4}{n+3}} = 1. \tag{1.4}$$

Therefore, (**b**) gives a better improvement of Theorem 1.1 for large n.

For relatively small n, (**a**) appears to be a better condition. For instance, the constant ζ in (**a**) is given by

$$\zeta \approx \begin{cases} 0.6581, & n = 3 \\ 0.6130, & n = 4 \\ 0.5774, & n = 5, \end{cases} \tag{1.5}$$

while $\cos\delta_0$ in (**b**) is restricted by (see by Lemma 2.3 (iii)),

$$\cos\delta_0 > \kappa \approx \begin{cases} 0.6919, & n = 3 \\ 0.6512, & n = 4 \\ 0.6155, & n = 5. \end{cases} \tag{1.6}$$

Thus, (**a**) provides a better improvement of Theorem 1.1 at least for dimensions $n = 3, 4, 5$.

Acknowledgment. The first author would like to thank Hubert Bray and Michael Eichmair for helpful discussions. The third author wants to thank Yuguang Shi for useful discussions.

2. Rigidity of geodesic balls

Throughout this paper, we let $\Omega = B(\delta) \subset \mathbb{S}^n$ be a (closed) geodesic ball of radius $\delta < \frac{\pi}{2}$, with boundary $\Sigma = \partial B(\delta)$. We denote by $\bar{g}$ the standard metric on $\mathbb{S}^n$, with volume form $d\mathrm{vol}_{\bar{g}}$ (*resp.* $d\sigma_{\bar{g}}$) on Ω (*resp.* Σ). We additionally define $\overline{\nabla}$ and $\Delta_{\bar{g}}$ to be the covariant derivative and Laplace operator of $\bar{g}$, and adopt the convention that the divergence, trace and norm (denoted by $\mathrm{div}(\cdot)$, $\mathrm{tr}(\cdot)$ and $|\cdot|$, respectively) are always computed with respect to $\bar{g}$.

We assume that $g = \bar{g} + h$ is a metric close to $\bar{g}$ (say $|h| \leq \frac{1}{2}$ at each point in Ω) and that g and $\bar{g}$ induce the same metric on Σ. The outward unit normal to Σ in $(\Omega, \bar{g})$ is denoted by $\overline{\nu}$, and X is the vector field on Σ dual to the 1-form $h(\cdot, \overline{\nu})|_{T(\Sigma)}$, *i.e.* $\bar{g}(v, X) = h(v, \overline{\nu})$ for any vector v tangent to Σ. Finally, for any function f and vector ν, $\partial_\nu f$ denotes the directional derivative of f along ν.

2.1. Brendle and Marques' proof.

The following weighted integral estimate of $(R(g) - R(\bar{g}))$ and $(H(g) - H(\bar{g}))$ plays a key role in the proof of Theorem 1.1 in [2].

Theorem 2.1 (Brendle and Marques [2]). *Let $\Omega = B(\delta)$ and $\lambda = \cos r$, where r is the $\bar{g}$-distance to the center of $B(\delta)$. Assume $\mathrm{div}(h) = 0$ where $h = g - \bar{g}$. Then*

$$
\begin{aligned}
&\int_\Omega [R(g) - n(n-1)]\lambda \, d\mathrm{vol}_{\bar{g}} + \int_\Sigma (2 - h(\overline{\nu}, \overline{\nu}))[H(g) - H(\bar{g})]\lambda \, d\sigma_{\bar{g}} \\
= &\int_\Omega \left[-\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}\, h)|^2) - \frac{1}{2}\left(|h|^2 + (\mathrm{tr}\, h)^2\right) \right] \lambda \, d\mathrm{vol}_{\bar{g}} \\
&+ \int_\Sigma H(\bar{g}) \left[-\frac{1}{4} h(\overline{\nu}, \overline{\nu})^2 - \frac{n}{2(n-1)}|X|^2 \right] \lambda \, d\sigma_{\bar{g}} \\
&+ \int_\Sigma \left[-h(\overline{\nu}, \overline{\nu})^2 - \frac{1}{2}|X|^2 \right] \partial_{\overline{\nu}} \lambda \, d\sigma_{\bar{g}} + \int_\Omega E(h) \, d\mathrm{vol}_{\bar{g}} + \int_\Sigma F(h) \, d\sigma_{\bar{g}}
\end{aligned}
$$

where $|E(h)| \leq C(|h|^3 + |\overline{\nabla} h|^3)$, $|F(h)| \leq C\left(|h|^3 + |h|^2 |\overline{\nabla} h|\right)$ for some constant C depending only on n.

To see how Theorem 1.1 follows from Theorem 2.1, one first pulls back g through a diffeomorphism $\varphi\colon \Omega \to \Omega$ with $\varphi|_\Sigma = \mathrm{id}$ such that $\varphi^*(g) - \bar{g}$ is $\bar{g}$-divergence free and $||\varphi^*(g) - \bar{g}||_{W^{2,p}(\Omega)} \leq N ||g - \bar{g}||_{W^{2,p}(\Omega)}$ for some $p > n$ and N depending only on Ω ([2, Proposition 11]).

Replacing g by $\varphi^*(g)$, one assumes that $\mathrm{div}(h) = 0$, where $h = g - \bar{g}$ and $||h||_{W^{2,p}(\Omega)}$ is small. If $R(g) \geq n(n-1)$ and $H(g) \geq H(\bar{g})$, Theorem 2.1 then implies

(2.1)
$$\begin{aligned}
&\int_\Omega \left[\frac{1}{4}(|\overline{\nabla}h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right] \lambda \, d\mathrm{vol}_{\bar{g}} \\
&+ \int_\Sigma h(\overline{\nu}, \overline{\nu})^2 \left[\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda\right] + |X|^2 \left[\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda\right] d\sigma_{\bar{g}} \\
\leq{}& C||h||_{C^1(\bar{\Omega})} \int_\Omega \left(|\overline{\nabla}h|^2 + |h|^2\right) d\mathrm{vol}_{\bar{g}}
\end{aligned}$$

for a constant C independent on h. At Σ, direct calculation shows

$$\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda = \frac{(n+3)\cos^2\delta - 4}{4\sin\delta} \tag{2.2}$$

$$\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda = \frac{(n+1)\cos^2\delta - 1}{2\sin\delta}. \tag{2.3}$$

If $\cos\delta \geq \frac{2}{\sqrt{n+3}}$, then both quantities in (2.2) and (2.3) are nonnegative. Therefore, (2.1) implies $h = 0$ if $||h||_{C^1(\bar{\Omega})}$ is sufficiently small.

2.2. **Improvement of Theorem 1.1: approach 1.** Let λ and h be given as in Theorem 2.1. Define

(2.4)
$$\begin{aligned}
W(h) = &\int_\Omega \left[\frac{1}{4}(|\overline{\nabla}h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right] \lambda \, d\mathrm{vol}_{\bar{g}} \\
&+ \int_\Sigma h(\overline{\nu}, \overline{\nu})^2 \left[\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda\right] + |X|^2 \left[\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda\right] d\sigma_{\bar{g}}.
\end{aligned}$$

It is clear from the above Brendle and Marques' proof that Theorem 1.1 holds on a geodesic ball $\Omega = B(\delta)$ provided one can prove

$$W(h) \geq \epsilon \int_\Omega \left(|\overline{\nabla}h|^2 + |h|^2\right) d\mathrm{vol}_{\bar{g}} \tag{2.5}$$

for some positive ϵ independent on h. To show (2.5), the difficulty lies in handling the boundary integral

$$\int_\Sigma h(\overline{\nu}, \overline{\nu})^2 \left[\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda\right] + |X|^2 \left[\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda\right] d\sigma_{\bar{g}}$$

which can be negative if $\cos\delta$ is small.

Proposition 2.1. *Let h be any C^2 symmetric (0,2) tensor on $\Omega = B(\delta)$ with $\operatorname{div}(h) = 0$. Let $c = \cos\delta$ and $s = \sin\delta$. Given any positive function w on Ω, we have*

$$s\int_\Sigma (\operatorname{tr} h) h(\overline{\nu},\overline{\nu}) d\sigma_{\bar g} \le \int_\Omega \left[\frac{w}{2}\sqrt{1-\lambda^2}|h|^2 + \lambda(\operatorname{tr} h)^2 + \frac{1}{2w}\sqrt{1-\lambda^2}|\overline{\nabla}(\operatorname{tr} h)|^2\right] d\mathrm{vol}_{\bar g}. \tag{2.6}$$

In particular, if $h|_{T(\Sigma)} = 0$, then

$$s\int_\Sigma h(\overline{\nu},\overline{\nu})^2 d\sigma_{\bar g} \le \int_\Omega \left[\frac{w}{2}\sqrt{1-\lambda^2}|h|^2 + \lambda(\operatorname{tr} h)^2 + \frac{1}{2w}\sqrt{1-\lambda^2}|\overline{\nabla}(\operatorname{tr} h)|^2\right] d\mathrm{vol}_{\bar g}. \tag{2.7}$$

Proof. Let ω be the 1-form on Ω given by

$$\omega_k = (\operatorname{tr} h) h_{ik}\overline{\nabla}^i\lambda.$$

Using the fact $\overline{\nabla}_k\overline{\nabla}^i\lambda = -\lambda\delta_k^i$ and the assumption $\operatorname{div}(h) = 0$, we have

$$\overline{\nabla}^k\omega_k = -\lambda(\operatorname{tr} h)^2 + h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h)).$$

At Σ, $\omega(\overline{\nu}) = -s(\operatorname{tr} h)h(\overline{\nu},\overline{\nu})$. It follows from the divergence theorem

$$s\int_\Sigma (\operatorname{tr} h) h(\overline{\nu},\overline{\nu}) d\sigma_{\bar g} = \int_\Omega \left[\lambda(\operatorname{tr} h)^2 - h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h))\right] d\mathrm{vol}_{\bar g}. \tag{2.8}$$

Given any positive function w on Ω, using the fact $|\overline{\nabla}\lambda|^2 = 1-\lambda^2$, we have

$$\begin{aligned} -h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h)) \le& |\overline{\nabla}\lambda|\,|h|\,|\overline{\nabla}(\operatorname{tr} h)| \\ \le& \sqrt{1-\lambda^2}\left[\frac{w}{2}|h|^2 + \frac{1}{2w}|\overline{\nabla}(\operatorname{tr} h)|^2\right]. \end{aligned} \tag{2.9}$$

Thus, (2.6) follows from (2.8) and (2.9). If $h|_{T(\Sigma)} = 0$, $h(\overline{\nu},\overline{\nu}) = \operatorname{tr} h$ at Σ. Therefore, (2.6) implies (2.7). □

Theorem 2.2. *Let δ be a constant in $(0, \frac{\pi}{2})$. Suppose $\cos\delta > \zeta$, where ζ is the positive constant given by*

$$\zeta^2 = \begin{cases} \frac{2}{n+1} & \text{if } n \le 4 \\ \frac{4(n+4)-4\sqrt{2n-1}}{n^2+6n+17} & \text{if } n \ge 5. \end{cases} \tag{2.10}$$

Then the conclusion of Theorem 1.1 holds on $B(\delta)$.

Proof. Let $c = \cos\delta$. Note that (2.10) implies $c^2 \ge \frac{1}{n+1}$, hence the coefficient of $|X|^2$ in (2.4) is nonnegative. By Theorem 1.1, it suffices

to assume $c^2 < \frac{4}{n+3}$. Choosing $w = \sqrt{2}$ in Proposition 2.1, we have

(2.11)
$$\begin{aligned} W(h) \geq & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right] \lambda \, d\mathrm{vol}_{\bar{g}} \\ & + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)} \int_\Omega \left(\frac{1}{2}|h|^2 + \frac{1}{4}|\overline{\nabla}(\operatorname{tr} h)|^2\right) d\mathrm{vol}_{\bar{g}} \\ & + \frac{(n+3)c^2 - 4}{4(1-c^2)} \int_\Omega \lambda (\operatorname{tr} h)^2 d\mathrm{vol}_{\bar{g}}. \end{aligned}$$

We seek conditions on c such that

$$c + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)} > 0 \tag{2.12}$$

and

$$\frac{1}{2} + \frac{(n+3)c^2 - 4}{4(1-c^2)} \geq 0. \tag{2.13}$$

Direct calculation shows that (2.12) (under the assumption $c^2 < \frac{4}{n+3}$) is equivalent to

$$c^2 > \frac{4(n+4) - 4\sqrt{2n-1}}{n^2 + 6n + 17} \tag{2.14}$$

and (2.13) is equivalent to

$$c^2 \geq \frac{2}{n+1}. \tag{2.15}$$

Since

$$\frac{4(n+4) - 4\sqrt{2n-1}}{n^2 + 6n + 17} \geq \frac{2}{n+1} \tag{2.16}$$

precisely when $n \geq 5$, we conclude that (2.5) holds for some $\epsilon > 0$ if (2.10) is satisfied. Theorem 2.2 is proved. $\square$

Theorem 2.2 verifies condition (**a**) in the introduction for $n \geq 5$. The remaining case $n = 3, 4$ in condition (**a**) will be verified in section 2.4.

2.3. **Improvement of Theorem 1.1: approach 2.** In this section, we give a different approach to estimate the boundary integral of $(\operatorname{tr} h)^2$ in $W(h)$ in terms of the interior integral in $W(h)$. To do so, we use the linearization of the scalar curvature (2.17). Noticing that the integral of $\operatorname{tr} h$ over $B(\delta)$ is close to zero, we apply the Poincaré inequality through an estimate of the first nonzero Neumann eigenvalue of $B(\delta)$ in [5].

Lemma 2.1. *Let $\Omega \subset \mathbb{S}^n$ be a closed domain with smooth boundary Σ. Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$ and $g = \bar{g} + h$ be another smooth metric on Ω such that g, $\bar{g}$ induce the same metric on Σ and $\operatorname{div} h = 0$. Suppose $|h|$ is very small, say $|h| \le \frac{1}{2}$ at every point.*

(i) *Given any smooth function f on Ω, one has*

$$\begin{aligned}&\int_\Omega f(\operatorname{tr} h)\Delta_{\bar{g}}(\operatorname{tr} h) + (n-1) f(\operatorname{tr} h)^2 \, d\mathrm{vol}_{\bar{g}} \\ &= \int_\Omega f(\operatorname{tr} h)\left[R(\bar{g}) - R(g)\right] d\mathrm{vol}_{\bar{g}} + E(h,f)\end{aligned}$$

where

$$|E(h,f)| \le C||f||_{C^1(\overline{\Omega})} \left(\int_\Omega \left(|h|^3 + |\overline{\nabla} h|^3\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2 |\overline{\nabla} h| d\sigma_{\bar{g}} \right)$$

for a positive constant C depending only on $(\Omega, \bar{g})$.

(ii)

$$\begin{aligned}\int_\Omega (\operatorname{tr} h) d\mathrm{vol}_{\bar{g}} = -\frac{1}{n-1} &\left(\int_\Omega \left[R(g) - R(\bar{g})\right] d\mathrm{vol}_{\bar{g}} \right. \\ &\left. + 2\int_\Sigma \left[H(g) - H(\bar{g})\right] d\sigma_{\bar{g}} \right) + F(h)\end{aligned}$$

where

$$|F(h)| \le C \left(\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma (|h|^2 + |h||\overline{\nabla} h|) d\sigma_{\bar{g}} \right)$$

for a positive constant C depending only on $(\Omega, \bar{g})$.

Proof. Since $\operatorname{div}(h) = 0$ and $\operatorname{Ric}(\bar{g}) = (n-1)\bar{g}$, h satisfies

$$-\Delta_{\bar{g}}(\operatorname{tr} h) - (n-1)(\operatorname{tr} h) = DR_{\bar{g}}(h), \tag{2.17}$$

where $DR_{\bar{g}}(\cdot)$ denotes the linearization of the scalar curvature at $\bar{g}$. By [2, Proposition 4] (also see [5, Lemma 2.1]), one knows

$$\begin{aligned}R(g) - R(\bar{g}) = {} & DR_{\bar{g}}(h) - \frac{1}{2} DR_{\bar{g}}(h^2) + \langle h, \overline{\nabla}^2 (\operatorname{tr} h)\rangle \\ & - \frac{1}{4}\left(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr}_{\bar{g}} h)|^2\right) + \frac{1}{2} h^{ij} h^{kl} \overline{R}_{ikjl} \\ & + E(h) + \overline{\nabla}_i (E_1^i(h))\end{aligned} \tag{2.18}$$

where $E(h)$ is a function and $E_1(h)$ is a vector field on Ω satisfying

$$|E(h)| \le C(|h||\overline{\nabla} h|^2 + |h|^3), \quad |E_1(h)| \le C|h|^2 |\overline{\nabla} h|$$

for a positive constant C depending only on n. Multiplying (2.17) by $f(\operatorname{tr} h)$ and integrating by parts, (i) follows from (2.18).

To prove (ii), we integrate (2.17) on Ω to get

$$-(n-1)\int_\Omega (\operatorname{tr} h) d\mathrm{vol}_{\bar g} = \int_\Omega DR_{\bar g}(h) d\mathrm{vol}_{\bar g} + \int_\Sigma \partial_{\overline{\nu}}(\operatorname{tr} h)\, d\sigma_{\bar g}. \tag{2.19}$$

Let $DH_{\bar g}(h)$ denote the linearization of the mean curvature of Σ at $\bar g$. Direct calculation (see [2, Proposition 5] or [4, (34)]) shows

$$2DH_{\bar g}(h) = \partial_{\overline{\nu}}(\operatorname{tr} h) - \operatorname{div} h(\overline{\nu}) - \operatorname{div}_\Sigma X. \tag{2.20}$$

Since $\operatorname{div}(h) = 0$, (2.20) implies

$$\int_\Sigma \partial_{\overline{\nu}}(\operatorname{tr} h)\, d\sigma_{\bar g} = 2\int_\Sigma DH_{\bar g}(h) d\sigma_{\bar g}. \tag{2.21}$$

By [2, Proposition 5], one has

$$|H(g) - H(\bar g) - DH_{\bar g}(h)| \le C(|h|^2 + |h||\overline{\nabla} h|) \tag{2.22}$$

for a positive constant C depending only on n. (ii) now follows from (2.18)-(2.22) and integration by parts on Ω. $\square$

We will make use of the first nonzero Neumann eigenvalue of $B(\delta)$, which we denote by $\mu(\delta)$. The next lemma on $\mu(\delta)$ was proved in [5, Lemma 3.1].

Lemma 2.2 ([5]). *Let $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$ (with respect to $\bar g$). Then*

(i) *$\mu(\delta)$ is a strictly decreasing function of δ on $(0, \frac{\pi}{2}]$;*
(ii) *for any $0 < \delta < \frac{\pi}{2}$,*

$$\mu(\delta) > n + \frac{(\sin\delta)^{n-2}\cos\delta}{\int_0^\delta (\sin t)^{n-1} dt} > \frac{n}{(\sin\delta)^2}.$$

Using $\mu(\delta)$, we have the following estimate of $\int_\Sigma (\operatorname{tr} h)^2 d\sigma_{\bar g}$.

Proposition 2.2. *Let $\Omega = B(\delta)$ and $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$. Let $g = \bar g + h$ be a smooth metric on $B(\delta)$ such that g, $\bar g$ induce the same metric on Σ and $\operatorname{div}(h) = 0$. Suppose $|h|$ is*

small, say $|h| \le \frac{1}{2}$ *at every point. Let* $c = \cos\delta$ *and* $s = \sin\delta$. *Then*

$$\begin{aligned} s\int_\Sigma (\operatorname{tr} h)^2 d\sigma_{\bar g} \le 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right]&\int_\Omega \lambda|\overline\nabla(\operatorname{tr} h)|^2 \ d\mathrm{vol}_{\bar g} \\ &- 2\int_\Omega (\lambda - c)(\operatorname{tr} h)(R(g) - R(\bar g)) \ d\mathrm{vol}_{\bar g} \\ &+ C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline\nabla h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2 \ d\sigma_{\bar g}\right] \\ &+ C\left[\int_\Omega (R(g) - R(\bar g)) \ d\mathrm{vol}_{\bar g} + 2\int_\Sigma (H(g) - H(\bar g)) \ d\sigma_{\bar g}\right]^2 \end{aligned}$$

for some positive constant C *depending only on* $(\Omega, \bar g)$ *and* c.

Proof. Integrating by parts, using the fact $\lambda = c$ at Σ and $\Delta_{\bar g}\lambda = -n\lambda$ on Ω, we have

(2.23)

$$\begin{aligned} \int_\Sigma (\operatorname{tr} h)^2 \partial_{\bar\nu}\lambda \ d\sigma_{\bar g} &= \int_\Omega (\operatorname{tr} h)^2 \Delta_{\bar g}\lambda - (\lambda - c)\Delta_{\bar g}(\operatorname{tr} h)^2 \ d\mathrm{vol}_{\bar g} \\ &= \int_\Omega -n\lambda(\operatorname{tr} h)^2 - 2(\lambda - c)[(\operatorname{tr} h)\Delta_{\bar g}(\operatorname{tr} h) + |\overline\nabla(\operatorname{tr} h)|^2] d\mathrm{vol}_{\bar g}. \end{aligned}$$

Choosing $f = \lambda - c$ in Lemma 2.1(i), we have

(2.24)

$$\begin{aligned} &\int_\Omega (\lambda - c)(\operatorname{tr} h)\Delta_{\bar g}(\operatorname{tr} h) \ d\mathrm{vol}_{\bar g} \\ = &\int_\Omega -(n-1)(\lambda - c)(\operatorname{tr} h)^2 - (\lambda - c)(\operatorname{tr} h)\left[R(g) - R(\bar g)\right] d\mathrm{vol}_{\bar g} + E_2(h) \end{aligned}$$

where

$$|E_2(h)| \le C\left(\int_\Omega \left(|h|^3 + |\overline\nabla h|^3\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2|\overline\nabla h| d\sigma_{\bar g}\right)$$

for some constant C depending on $(\Omega, \bar g)$ and c. It follows from (2.23) and (2.24) that

(2.25)

$$\begin{aligned} \int_\Sigma (\operatorname{tr} h)^2 \partial_{\bar\nu}\lambda \ d\sigma_{\bar g} = &\int_\Omega \left[(n-2)(\operatorname{tr} h)^2 - 2|\overline\nabla(\operatorname{tr} h)|^2\right] \lambda \ d\mathrm{vol}_{\bar g} \\ &+ 2c\int_\Omega \left[|\overline\nabla(\operatorname{tr} h)|^2 - (n-1)(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g} \\ &+ 2\int_\Omega (\lambda - c)(\operatorname{tr} h)\left[R(g) - R(\bar g)\right] d\mathrm{vol}_{\bar g} - 2E_2(h). \end{aligned}$$

Since $\lambda \geq c$ on Ω, (2.25) implies

$$\int_\Sigma (\operatorname{tr} h)^2 \partial_{\bar\nu}\lambda \, d\sigma_{\bar g} \geq -2\int_\Omega |\overline\nabla(\operatorname{tr} h)|^2 \lambda d\mathrm{vol}_{\bar g} + 2c\int_\Omega \left[|\overline\nabla(\operatorname{tr} h)|^2 - \frac{n}{2}(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g}$$
$$+ 2\int_\Omega (\lambda - c)(\operatorname{tr} h)\left[R(g) - R(\bar g)\right] d\mathrm{vol}_{\bar g} - 2E_2(h).$$

By the variational characterization of $\mu(\delta)$, we have

(2.26)
$$\int_\Omega |\overline\nabla(\operatorname{tr} h)|^2 \, d\mathrm{vol}_{\bar g} \geq \mu(\delta)\left[\left(\int_\Omega (\operatorname{tr} h)^2 \, d\mathrm{vol}_{\bar g}\right) - \frac{1}{V(\bar g)}\left(\int_\Omega (\operatorname{tr} h) \, d\mathrm{vol}_{\bar g}\right)^2\right]$$

where $V(\bar g) = \int_\Omega 1 d\mathrm{vol}_{\bar g}$. It follows from Lemma 2.1(ii) and (2.26) that

$$\begin{aligned} &\int_\Omega \left[|\overline\nabla(\operatorname{tr} h)|^2 - \frac{n}{2}(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g} \\ &\geq \left(1 - \frac{n}{2\mu(\delta)}\right)\int_\Omega |\overline\nabla(\operatorname{tr} h)|^2 \, d\mathrm{vol}_{\bar g} \\ &\quad - C\left[\int_\Omega (R(g) - R(\bar g)) \, d\mathrm{vol}_{\bar g} + 2\int_\Sigma (H(g) - H(\bar g)) \, d\sigma_{\bar g}\right]^2 \\ &\quad - C\left[\int_\Omega \left(|h|^2 + |\overline\nabla h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma (|h|^2 + |h||\overline\nabla h| d\sigma_{\bar g})\right]^2 \end{aligned} \tag{2.27}$$

for a positive constant C depending only on $(\Omega, \bar g)$. The lemma now follows from (2.25), (2.27) and the fact $\lambda \leq 1$. □

The following lemma is needed for the statement of Theorem 2.3.

Lemma 2.3. *On $(0, \frac{\pi}{2}]$, define*

$$\alpha(\delta) = \left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1}\frac{(n+1)}{8n}$$

and

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}.$$

Then

(i) *$\alpha(\delta)$ is strictly decreasing, $\lim_{\delta\to 0+}\alpha(\delta) = \infty$ and $\alpha(\frac{\pi}{2}) = \frac{n+1}{8n}$.*
(ii) *$F(\delta)$ is strictly decreasing, $\lim_{\delta\to 0+} F(\delta) = \infty$ and $F(\frac{\pi}{2}) < 0$. Hence there is exactly one $\delta_0 \in (0, \frac{\pi}{2})$ such that $F(\delta_0) = 0$.*
(iii) *$\cos\delta_0 > \kappa$ where κ is the positive root of the equation*

$$2n(n+3)x^2 + (n+1)x + (1 - 7n) = 0.$$

In particular, $(\cos\delta_0)^2 > \frac{1}{n+1}$.

Proof. (i) follows directly from Lemma 2.2. (ii) follows from (i) and the fact

$$F(\delta) = \alpha(\delta) + \frac{n-1}{4}\frac{1}{\sin^2\delta} - \frac{n+3}{4}.$$

To prove (iii), suppose $\cos\delta_0 = a$. Since $0 < 1 - \frac{n}{2\mu(\delta_0)} < 1$, one has $\left(1 - \frac{n}{2\mu(\delta_0)}\right)\cos\delta_0 < a$ and $\alpha(\delta_0) < \frac{n+1}{8n}\frac{1}{(1-a)}$. Therefore,

$$0 = F(\delta_0) < \frac{n+1}{8n}\frac{1}{(1-a)} + \frac{n-1}{4}\frac{1}{1-a^2} - \frac{n+3}{4}$$

which implies (iii). □

Theorem 2.3. *Let $\Omega = B(\delta)$ be a geodesic ball of radius δ in $\mathbb{S}^n$. Suppose $\delta < \delta_0$, where δ_0 is the unique zero in $(0, \frac{\pi}{2})$ of the function*

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}$$

where $\alpha(\delta) = \left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1}\frac{(n+1)}{8n}$. Then the conclusion of Theorem 1.1 holds on Ω.

Proof. Let $W(h)$ be given in (2.4). Let $c = \cos\delta$. Lemma 2.3(iii) shows $c^2 > \frac{1}{n+1}$. Hence, the coefficient of $|X|^2$ in $W(h)$ is nonnegative. By Theorem 1.1, it suffices to assume $c^2 < \frac{4}{n+3}$. Apply Proposition 2.2, we have

(2.28)
$$\begin{aligned} W(h) \geq & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right]\lambda \, d\mathrm{vol}_{\bar g} \\ & + \left[\frac{(n+3)c^2 - 4}{4(1-c^2)}\right] 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right]\int_\Omega |\overline{\nabla}(\operatorname{tr} h)|^2 \lambda \, d\mathrm{vol}_{\bar g} \\ & + \hat{E}(h, c), \end{aligned}$$

where

(2.29)
$$\begin{aligned} \hat{E}(h,c) = & \left[\frac{(n+3)c^2 - 4}{4(1-c^2)}\right]\Bigg\{-2\int_\Omega (\lambda - c)(\operatorname{tr} h)(R(g) - R(\bar g)) d\mathrm{vol}_{\bar g} \\ & + C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2 \, d\sigma_{\bar g}\right] \\ & + C\left[\int_\Omega (R(g) - R(\bar g)) \, d\mathrm{vol}_{\bar g} + 2\int_\Sigma (H(g) - H(\bar g)) \, d\sigma_{\bar g}\right]^2\Bigg\}. \end{aligned}$$

Since $\delta < \delta_0$, Lemma 2.3 (ii) implies

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} > F(\delta_0) = 0.$$

Hence there exists a small constant $\epsilon \in (0,1)$ such that

$$\frac{1}{4}\left(1 + \frac{(1-\epsilon)}{n}\right) + \left[\frac{(n+3)c^2 - 4}{4(1-c^2)}\right] 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right] > 0. \tag{2.30}$$

By (2.28) and (2.30), using the fact $|\overline{\nabla} h|^2 \geq \frac{1}{n}|\overline{\nabla}(\operatorname{tr} h)|^2$, we have

$$W(h) \geq \frac{1}{4}\epsilon c \int_\Omega (|\overline{\nabla} h|^2 + |h|^2)\, d\mathrm{vol}_{\bar g} + \hat{E}(h,c). \tag{2.31}$$

Now suppose $R(g) - R(\bar g) \geq 0$, $H(g) - H(\bar g) \geq 0$ and $||h||_{W^{2,p}(\Omega)}$ is sufficiently small. It follows from Theorem 2.1, (2.29) and (2.31) that

$$\begin{aligned} &\frac{1}{2}\int_\Omega [R(g) - R(\bar g)]\lambda\, d\mathrm{vol}_{\bar g} + \frac{1}{2}\int_\Sigma [H(g) - H(\bar g)]\lambda\, d\sigma_{\bar g} \\ &\leq \epsilon \int_\Omega (|\overline{\nabla} h|^2 + |h|^2)\, d\mathrm{vol}_{\bar g} \\ &\quad + C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2\, d\sigma_{\bar g}\right]. \end{aligned} \tag{2.32}$$

for some positive constant C independent of h. We can then proceed as in [2]: since $||h||_{L^2(\Sigma)} \leq C||h||_{W^{1,2}(\Omega)}$, one knows the terms in the last line in (2.32) is bounded by $C||h||_{C^1(\overline{\Omega})}||h||_{W^{1,2}(\Omega)}$. Therefore, if $||h||_{W^{2,p}(\Omega)}$ is sufficiently small, (2.32) implies h must vanish identically. This completes the proof of Theorem 2.3. $\square$

We give some lower estimates of δ_0 which are relatively more explicit.

Proposition 2.3. *δ_0 in Theorem 2.3 satisfies*

(i) $\delta_0 > \tilde\delta_0$ *where $\tilde\delta_0$ is the unique zero in $(0, \frac{\pi}{2})$ of the equation*

$$\left[1 - \left(1 - \frac{n}{2\tilde\mu(\delta)}\right)\cos\delta\right]^{-1}\frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} = 0$$

where $\tilde\mu(\delta) = n + \dfrac{(\sin\delta)^{n-2}\cos\delta}{\int_0^\delta (\sin t)^{n-1}dt}$.

(ii) $\cos\delta_0 < \tilde\kappa$ *where $\tilde\kappa$ is the unique zero in $(0,1)$ of the equation*

$$n(n+3)x^4 + n(n+3)x^3 + 2n(n+1)x^2 + (1-3n)x - 7n + 1 = 0.$$

(iii) $(\cos\delta_0)^2 < \dfrac{7n-1}{2n^2+5n-1}$.

Proof. By Lemma 2.2 (ii), $\mu(\delta_0) > \tilde{\mu}(\delta_0)$. Hence,

$$\left[1 - \left(1 - \frac{n}{2\tilde{\mu}(\delta_0)}\right)\cos\delta_0\right]^{-1} \frac{n+1}{8n} + \frac{(n+3)\cos^2\delta_0 - 4}{4(1-\cos^2\delta_0)} < 0. \tag{2.33}$$

Note that $\tilde{\mu}(\delta)$ is strictly decreasing in $(0, \frac{\pi}{2}]$. As in the proof of Lemma 2.3(ii), we know the function

$$\left[1 - \left(1 - \frac{n}{2\tilde{\mu}(\delta)}\right)\cos\delta\right]^{-1} \frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)}$$

is strictly decreasing and has a unique zero $\tilde{\delta}_0$ in $(0, \frac{\pi}{2})$. Hence, (i) follows from (2.33).

The proof of (ii) is similar to that of (i) except we replace the lower bound $\mu(\delta) > \tilde{\mu}(\delta)$ by a weaker lower bound $\mu(\delta_0) > \frac{n}{(\sin\delta_0)^2} = \frac{n}{1-(\cos\delta_0)^2}$.

(iii) follows from the fact

$$\frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} < 0.$$

□

Theorem 2.3 and Proposition 2.3 (iii) verify condition (**b**) in the introduction.

2.4. **A Combined approach.** It remains to confirm the case $n = 3, 4$ in condition (**a**). To do so, we combine the two methods leading to Theorem 2.2 and Theorem 2.3.

Theorem 2.4. *Suppose* $3 \leq n \leq 4$, *Theorem 1.1 is true on* $B(\delta)$ *if*

$$\cos\delta > \left(\frac{4(n+4) - 4\sqrt{2n-1}}{n^2+6n+17}\right)^{\frac{1}{2}} \approx \begin{cases} 0.6581, & n=3 \\ 0.6130, & n=4. \end{cases} \tag{2.34}$$

Proof. Let $c = \cos\delta$. (2.34) implies $c^2 > \frac{1}{n+1}$. By (2.11), we have $W(h) \geq Y(h)$ where

$$\begin{aligned} Y(h) = &\left[c + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)}\right] \int_\Omega \left(\frac{1}{2}|h|^2 + \frac{1}{4}|\overline{\nabla}(\operatorname{tr} h)|^2\right) d\mathrm{vol}_{\bar{g}} \\ &+ \left[\frac{1}{2} + \frac{(n+3)c^2 - 4}{4(1-c^2)}\right] \int_\Omega \lambda(\operatorname{tr} h)^2 d\mathrm{vol}_{\bar{g}} + \frac{c}{4}\int_\Omega |\overline{\nabla} h|^2 \, d\mathrm{vol}_{\bar{g}}. \end{aligned}$$

Now (2.34) implies (2.12), i.e.

$$c + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)} > 0. \tag{2.35}$$

To continue, we only need to assume $\frac{1}{2}+\frac{(n+3)c^2-4}{4(1-c^2)}<0$. (If $n\geq 5$, this term would be nonnegative by (2.16).)

Given any constants $\theta,\tau\in(0,1)$, using the fact $|\overline{\nabla}h|^2\geq\frac{1}{n}|\overline{\nabla}(\operatorname{tr}h)|^2$, $|h|^2\geq\frac{1}{n}(\operatorname{tr}h)^2$, $\lambda\leq 1$ and applying (2.26) as in Theorem 2.3, we have

(2.36)

$$\begin{aligned}
Y(h)\geq&\int_\Omega\left\{\frac{\theta c}{4}|\overline{\nabla}h|^2+\frac{1}{4}\left[\frac{1-\theta}{n}c+c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]|\overline{\nabla}(\operatorname{tr}h)|^2\right.\\
&+\tau\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{|h|^2}{2}+\frac{1-\tau}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{(\operatorname{tr}h)^2}{2}\\
&\left.+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right]\frac{(\operatorname{tr}h)^2}{2}\right\}\,d\mathrm{vol}_{\bar g}\\
\geq&\,\epsilon\left(\int_\Omega|\overline{\nabla}h|^2+|h|^2d\mathrm{vol}_{\bar g}\right)\\
&+\left\{\frac{1}{2}\left[\frac{(n+1)-\theta}{n}c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\mu(\delta)+\frac{1-\tau}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\right.\\
&\left.+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right]\right\}\left(\int_\Omega\frac{(\operatorname{tr}h)^2}{2}\,d\mathrm{vol}_{\bar g}\right)+E(h)
\end{aligned}$$

where $\epsilon=\min\left\{\frac{\theta c}{4},\frac{\tau}{2}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\right\}>0$, $\mu(\delta)$ is the first nonzero Neumann eigenvalue of $B(\delta)$, and $E(h)$ is an error term satisfying

$$\begin{aligned}
|E(h)|\leq&C\left[\int_\Omega(R(g)-R(\bar g))\,d\mathrm{vol}_{\bar g}+2\int_\Sigma(H(g)-H(\bar g))\,d\sigma_{\bar g}\right]^2\\
&+C\left[\int_\Omega\left(|h|^2+|\overline{\nabla}h|^2\right)d\mathrm{vol}_{\bar g}+\int_\Sigma(|h|^2+|h||\overline{\nabla}h|)d\sigma_{\bar g}\right]^2
\end{aligned}$$

with C depending only on $B(\delta)$.

Apply the eigenvalue estimate $\mu(\delta)>\frac{n}{(\sin\delta)^2}=\frac{n}{1-c^2}$ (Lemma 2.2 (ii)), one checks (using *Mathematica*) that

$$\begin{aligned}
0<&\frac{1}{2}\left[\frac{n+1}{n}c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\mu(\delta)+\frac{1}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\\
&+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right]
\end{aligned}\tag{2.37}$$

for $1>c>0.6378$ when $n=3$ and for $1>c>0.5933$ when $n=4$. In particular, (2.37) is guaranteed by (2.34).

Therefore, there exist small positive constants θ, τ such that the coefficient of $\int_\Omega \frac{(\operatorname{tr} h)^2}{2} d\mathrm{vol}_{\bar g}$ in (2.36) is positive. For these θ and τ, we have

$$W(h) \geq Y(h) \geq \epsilon \left(\int_\Omega |\overline{\nabla} h|^2 + |h|^2 d\mathrm{vol}_{\bar g} \right) + E(h).$$

Arguing as in the proof of Theorem 2.3 (the part following (2.31)), we conclude that Theorem 1.1 holds on such a $B(\delta)$. $\square$

References

[1] Brendle, S., *Rigidity phenomena involving scalar curvature*, arXiv:1008.3097v2, to appear in Surveys in Differential Geometry.

[2] Brendle, S. and Marques, F. C., *Scalar curvature rigidity of geodesic balls in* $\mathbb{S}^n$, J. Differential Geom. **88** (2011), 379–394.

[3] Brendle, S., Marques, F. C. and Neves, A., Deformations of the hemisphere that increase scalar curvature, Invent. Math. **185** (2011), 175–197.

[4] Miao, P. and Tam, L.-F., *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. **36** (2009), 141-171.

[5] Miao, P. and Tam, L.-F., *Scalar curvature rigidity with a volume constraint*, arXiv:1109.2960v2, to appear in Comm. Anal. Geom.

[6] Min-Oo, M., *Scalar curvature rigidity of certain symmetric spaces*, Geometry, topology, and dynamics (Montreal, 1995), 127-137, CRM Proc. Lecture Notes vol. 15, Amer. Math. Soc., Providence RI, 1998

(Graham Cox) Department of Mathematics, Duke University, Durham, NC 27708, USA.

E-mail address: ghcox@math.duke.edu

(Pengzi Miao) School of Mathematical Sciences, Monash University, Victoria 3800, Australia; Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA.

E-mail address: Pengzi.Miao@sci.monash.edu.au; pengzim@math.miami.edu

(Luen-Fai Tam) The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

E-mail address: lftam@math.cuhk.edu.hk